

## DICTATOR LEMMA: DETAILED PROOF

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### *Abstract*

The purpose of this note is to put forward a self-contained proof of the Dictator Lemma, which was used in the simple alternative proof of Arrow's General Possibility Theorem (*Hitotsubashi Journal of Economics*, Vol. 29, 1988). It is hoped that this note will make my proof more accessible to the wider audience.

In my alternative proof [Suzumura (1988)] of Arrow's general possibility theorem, I made use of the following:<sup>1</sup>

**Dictator Lemma:** Let  $F_n$  be an SWF satisfying Pareto and Independence axioms, where  $n \geq 2$ . If there exist  $i \in I(n)$ ,  $a \in d(F_n)$  and  $x, y \in X$  such that

$$(1) \quad xP_i^a y \text{ \& } [yP_j^a x \text{ for all } j \in I(n) - \{i\}] \rightarrow xP^a y,$$

then  $i$  is a dictator for  $F_n$ .

It so happened that my proof of this lemma was too terse for somebody who are unaccustomed to be fully understood. Let me record my full proof for the sake of making my argument more accessible.

### Proof of Dictator Lemma

**Step 1:** Take any  $z \in X - \{x, y\}$  and let  $b \in d(F_n)$  be such that  $xP_i^b y$ ,  $yP_j^b x$  for all  $j \in I(n) - \{i\}$ , and  $yP_j^b z$  for all  $j \in I(n)$ . No specification is made of  $R_j^b (j \neq i)$  over  $\{x, z\}$ . By Independence and (1), we obtain  $xP^b y$ , whereas Pareto implies  $yP^b z$ . It follows that  $xP^b z$  by virtue of transitivity of  $P^b$ . Invoking Independence once again, we can assert that

$$(2) \quad \text{For all } a \in d(F_n), \text{ if } xP_i^a z, \text{ then } xP^a z.$$

For mnemonic convenience let me denote this by  $D_i(x, z)$ .

**Step 2:** Let  $b \in d(F_n)$  be such that  $xP_i^b z$  and  $zP_j^b y$  for all  $j \in I(n)$ . It follows from  $D_i(x, z)$  that  $xP^b z$ , whereas Pareto yields  $zP^b y$ , so that  $xP^b y$  follows from transitivity of  $P^b$ . Invoking Independence, we obtain  $D_i(x, y)$ .

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<sup>1</sup> Throughout this note, my notation will be the same as in Suzumura (1988).

Step 3: Let  $c \in d(F_n)$  be such that  $xP_i^c y$  and  $zP_j^c x$  for all  $j \in I(n)$ . It follows from  $D_i(x, y)$  [resp. Pareto] that  $xP^c y$  [resp.  $zP^c x$ ], so that  $zP^c y$  follows from transitivity of  $P^c$ . Invoking Independence, we obtain  $D_i(z, y)$ .

Step 4: Let  $d \in d(F_n)$  be such that  $zP_i^d y$  and  $yP_j^d x$  for all  $j \in I(n)$ . It follows from  $D_i(z, y)$  [resp. Pareto] that  $zP^d y$  [resp.  $yP^d x$ ], so that  $zP^d x$  follows from transitivity of  $P^d$ . Invoking Independence, we obtain  $D_i(z, x)$ .

Step 5: Let  $e \in d(F_n)$  be such that  $xP_i^e z$  and  $yP_j^e x$  for all  $j \in I(n)$ . It follows from  $D_i(x, z)$  [resp. Pareto] that  $xP^e z$  [resp.  $yP^e x$ ], so that  $yP^e z$  follows from transitivity of  $P^e$ . Invoking Independence, we obtain  $D_i(y, z)$ .

Step 6: Let  $f \in d(F_n)$  be such that  $yP_i^f z$  and  $zP_j^f x$ . It follows from  $D_i(y, z)$  [resp.  $D_i(z, x)$ ] that  $yP^f z$  [resp.  $zP^f x$ ], so that  $yP^f x$  follows from transitivity of  $P^f$ . Invoking Independence, we obtain  $D_i(y, x)$ .

Final Step: We have thus shown that (1) implies  $D_i(\alpha, \beta)$  for every ordered pair  $(\alpha, \beta)$  taken from  $\{x, y, z\}$ , so that  $i$  gets his way over  $\{x, y, z\}$  whatever others may say, viz.,  $i$  is a dictator over  $\{x, y, z\}$ . If  $\#X \geq 4$ , we may replace  $z$  by any  $w \in X - \{x, y, z\}$  to complete our proof that  $i$  is fully dictatorial over  $X$  if (1) holds.  $\square$

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## REFERENCES

- Arrow, K. J. (1963), *Social Choice and Individual Values*, 2nd ed., New York, Wiley.  
 Suzumura, K. (1988), "Reduction of Social Choice Problems: A Simple Proof of Arrow's General Possibility Theorem," *Hitotsubashi Journal of Economics*, Vol. 29, 1988, pp. 219-221.